# Sensitivity function and entropy increase rates for $z$-logistic map family at the edge of chaos

Ahmet Celikoglu and Ugur Tirnakli *

*Department of Physics, Faculty of Science, Ege University, 35100 Izmir, Turkey*

**Abstract**

It is well known that, for chaotic systems, the production of relevant entropy (Boltzmann-Gibbs) is always linear and the system has strong (exponential) sensitivity to initial conditions. In recent years, various numerical results indicate that basically the same type of behavior emerges at the edge of chaos if a specific generalization of the entropy and the exponential are used. In this work, we contribute to this scenario by numerically analysing some generalized nonextensive entropies and their related exponential definitions using $z$-logistic map family. We also corroborate our findings by testing them at accumulation points of different cycles.

*Key words:* Nonlinear dynamics, non-extensivity
*PACS:* 05.45.-a, 05.20.-y

## 1 Introduction

In nonlinear dynamics, it is well known that the chaotic systems has an exponential sensitivity to initial conditions, characterized by the sensitivity function (for one-dimensional case)

$$\xi(t) \equiv \lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)}, \tag{1}$$

(where $\Delta x(t)$ is the distance, in phase space, between two copies at time $t$) to diverge as $\xi(t) = \exp(\lambda_1 t)$, where $\lambda_1$ is the standard Lyapunov exponent [1]. On the other hand, for the marginal case, where $\lambda_1 = 0$, the form of the

* Corresponding author.
*Email address:* ugur.tirnakli@ege.edu.tr (Ugur Tirnakli).

sensitivity function could be a whole class of functions. One of the candidates is a power-law behavior, which can be characterized by an appropriate generalization of exponentials, namely $\xi(t) = \widetilde{\exp}(\lambda t)$, where $\lambda$ is the generalized Lyapunov exponent. Therefore, $\lambda > 0$ and $\lambda < 0$ cases correspond to weak sensitivity and weak insensitivity to initial conditions respectively. This, in fact, constitutes a unified framework since the generalized exponentials include the standard one as a special case for an appropriate choice of related parameter.

The other concept that we focus in this work is the entropy production. For a chaotic system, the Kolmogorov-Sinai (KS) entropy $K_1$ is defined as the increase, per unit time, of the standard Boltzmann-Gibbs entropy and it is basically related to the standard Lyapunov exponents through the Pesin identity, which states that $K_1 = \lambda_1$ if $\lambda_1 > 0$ and $K_1 = 0$ otherwise. Here, it is worth mentioning that the KS entropy is basically defined in terms of a single trajectory in phase space, using a symbolic representation of the regions of a partitioned phase space (see [1]). However, it appears that, in most cases, this definition can be replaced by one based on an ensemble of initial conditions, which is the version we use herein. In the framework of this version, it has already been shown that the statistical definition of entropy production rate exhibits a close analogy to the production rate of thermal entropy and practically coincides with the KS entropy in chaotic systems [2]. In recent years, there have been efforts on extending this picture to dynamical systems at their marginal points (like chaos threshold) by using a generalized entropic form, which allows us to define a generalized KS entropy $K$. From this definition, one can also conjecture a Pesin-like identity as $K = \lambda$, which recovers both the chaotic and critical (i.e., chaos thresholds) cases since it includes standard Pesin identity as a special case [3].

As a whole, this unified framework has been numerically verified firstly for the logistic map [3,4] using the Tsallis entropy $S_q \equiv \left(1 - \sum_{i=1}^{W} p_i^q\right) / (q-1)$ [5], which grows linearly for a special value of entropic index $q$, which is $q_{sen} \simeq 0.24$; whereas the asymptotic power-law sensitivity to initial conditions is characterized by the generalized exponential $\widetilde{\exp}(x) = \exp_q(x) = [1 + (1-q)x]^{1/(1-q)}$ with the same value of the entropic index. After these works on the logistic map, numerical evidences supporting this framework came also from the studies of other low-dimensional dynamical systems, such as the $z$-logistic map family [6,7], the Henon map [8] and the asymmetric logistic map family [9]. Besides these numerical investigations, analytical treatment of the subject is also available recently in a series of paper by Baldovin and Robledo [10]. Very recently, a similar analysis has been performed for the $z$-logistic map family, but this time, using ensemble-averaged initial conditions distributed uniformly over the entire available phase space [11,12]. The most important outcome of this analysis is another numerical verification of the coincidence of the entropic indices coming from the sensitivity function and entropy production rates (although with a different value $q_{sen}^{av} \simeq 0.36$), which consequently broadens the

validity of the Pesin-like identity.

Finally, here we should mention a recent work of Tonelli et al. [13], where they demonstrate that the above-mentioned framework is even more general by making use of a two-parameter family of logarithms [14]

$$\widetilde{\log}(\xi) = \frac{\xi^{\alpha} - \xi^{-\beta}}{\alpha + \beta} \tag{2}$$

where $\alpha$ ($\beta$) characterizes the large (small) argument asymptotic behavior. From this wide class, they analysed four interesting one-parameter cases, namely ;
(i) the Tsallis logarithm [5] : $\alpha = 1 - q$ and $\beta = 0$
(ii) the Abe logarithm [15] : $\alpha = 1 - q$ and $\beta = \alpha/(1 + \alpha)$
(iii) the Kaniadakis logarithm [16] : $\alpha = \beta = \kappa = 1 - q$
(iv) the $\gamma$ logarithm : $\alpha = 1 - q$ and $\beta = (1 - q)/2$.
Their analysis consists of studying the sensitivity function and the entropy increase rates for the logistic map at the edge of chaos. Obviously, for the corresponding entropy in each case, one needs to use

$$S(t) = \sum_{i=1}^{W} p_i(t) \widetilde{\log}\left(\frac{1}{p_i(t)}\right) \tag{3}$$

from where the entropy production rates in time can be calculated. Their numerical results clearly verified that, for the logistic map, the relavant value of the entropic index is $q_{sen}^{av} \simeq 0.36$ not only for the Tsallis case but also for the others as well, and moreover that the Pesin-like identity is also present for all cases (with different numerical values for each case).

The aim of the present effort is to check the validity of the above-mentioned picture making use of the $z$-logistic map family. Moreover, the effect of different cycles on this validity is tested by analysing four distinct cycles.

## 2 The model and the procedure

The model system that we use in our analysis is the $z$-logistic map family defined as

$$x_{t+1} = 1 - a|x_t|^z \tag{4}$$

where $z > 1$; $0 < a \leq 2$; $|x_t| \leq 1$; $t = 0, 1, 2, ....$ It is easily seen that $z = 2$ case corresponds to the standard logistic map.

Table 1
The critical $a_c$ values of the $z$-logistic map family for all cycles used in this study.

| $z$ | $a_c$ | | | |
|---|---|---|---|---|
| | cycle 2 | cycle 3 | cycle 4 | cycle 5 |
| 1.75 | 1.35506... | 1.74730... | 1.92764... | 1.60749... |
| 2 | 1.40115... | 1.77981... | 1.94217... | 1.63101... |
| 2.5 | 1.47054... | 1.82886... | 1.96144... | 1.66954... |
| 3 | 1.52187... | 1.86299... | 1.97302... | 1.69944... |
| 4 | 1.59490... | 1.90597... | 1.98524... | 1.74282... |

Firstly we study the sensitivity to initial conditions at the edge of chaos using the sensitivity function given in Eq.(1). From its definition, for the calculation, we proceed with considering two initially very close points, which makes for example $\Delta x(0) = 10^{-12}$ and then at each time step we numerically calculate the sensitivity function. To make an ensemble average, this procedure is repeated many times starting from different initial conditions all chosen randomly within the phase space and an average is taken over all values of $\widetilde{\log}(\xi)$. The special value of $\alpha$ (consequently, $q_{sen}^{av}$) is found as the value for which we obtain a linear time dependence of $\left\langle \widetilde{\log}(\xi) \right\rangle (t)$. Finally, it is clear that the slope of this curve gives us the generalized Lyapunov exponent $\lambda$. In our analysis we concentrate on various values of $z$ for four different cycles at their accumulation points denoted by $a_c$. These critical values are given in Table 1.

Secondly we study the entropy production rates of the entropies related to four selected logarithms, namely we use Eq.(3). The procedure that we implement for the entropy production is the same as the one introduced firstly in [2] for chaotic systems and then used in other works related to this context. It consists of dividing the phase space in $W$ equal intervals and putting randomly $N$ initial points in one of them. Then, one should trace the spread of initial points within the phase space and calculate the entropy $S(t)$, from where we can obtain the entropy production per unit time as

$$K = \lim_{t\to\infty} \lim_{W\to\infty} \lim_{N\to\infty} \frac{S(t)}{t}. \quad (5)$$

We then repeat this many times starting from randomly chosen different intervals of the phase space and average the entropy $S(t)$ over these experiments. Here, the special value of $\alpha$ (consequently, $q_{sen}^{av}$) is obviously found as the value for which we obtain a linear time dependence of $\langle S \rangle (t)$. In addition to this,

the value of $K$ can be obtained from the slope of this linear curve.

From the previous results [11,12], it is known for the Tsallis case that, for the $z$-logistic map family, the proper $\alpha$ values (namely, $q_{sen}^{av}$) coming from the sensitivity function and entropy production coincide each other and moreover, for all $z$ values, for the proper $\alpha$ value the Pesin-like identity is preserved as $K = \lambda$. Then, very recently, for the standard logistic map (namely, $z = 2$ case), Tonelli et al. [13] showed that the picture is wider that this by considering the other three cases mentioned above. Now, our aim is to check the validity of this framework for the $z$-logistic map and for various cycles.

## 3 Numerical results and conclusions

In our simulations, for the sensitivity function analysis, for each cycle and $z$ value we consider two very close points so that $\Delta x(0) = 10^{-12}$ and then numerically calculate the sensitivity function $\xi$ from its definition. In order to take the logarithm of $\xi$, we use four different cases mentioned previously. We repeat this for $4 \times 10^7$ uniformly distributed initial conditions to make an ensemble average of $\left\langle \widetilde{\log}(\xi) \right\rangle (t)$. Its time evolution is given for a representative value of $z$ for cycles 3, 4 and 5 in Fig.1(a),(c) and (e) respectively. For each case (namely, Tsallis, Abe, Kaniadakis and $\gamma$ cases), the special value of $\alpha$ is determined by testing various $\alpha$ values until we obtain a linear time dependence. From the slope of each curve, the corresponding generalized Lyapunov exponents $\lambda$ can be calculated. All of the obtained results for $\alpha$ and $\lambda$ are given in Table 2. For a given $z$ value of each cycle, it is seen that the special value of $\alpha$ is the same (within error bars) for all four generalized cases. However, generically the generalized Lyapunov exponents are different in each case.

Then, we study the entropy increase rates. We divide the phase space into $W$ equal cells and put $N$ initial conditions into a randomly chosen cell. We then let the dynamics evolve in time and calculate the generalized entropies. In order to make an ensemble average, we repeat this procedure many times starting from randomly chosen different cells ($W/2$ cells), which reduces also the fluctuations that appear at the edge of chaos. In all our simulations we use $W = 10^5$ and $N = 10W$. For each entropic form, we use the special value of $\alpha$ calculated from the sensitivity function and we obtain the linear coefficient $K$ from the slope of the linear entropy increase rates as it is seen in Fig.1(b),(d) and (f). As it is clear from Table 2 that, for all $z$ values of each cycle and for all entropic forms, $K$ values coincide with corresponding $\lambda$ values (within numerical errors).

Finally, it is worth mentioning that generically for the increasing values of $z$ (depending also on the cycle), the observed fluctuations in the time evolution

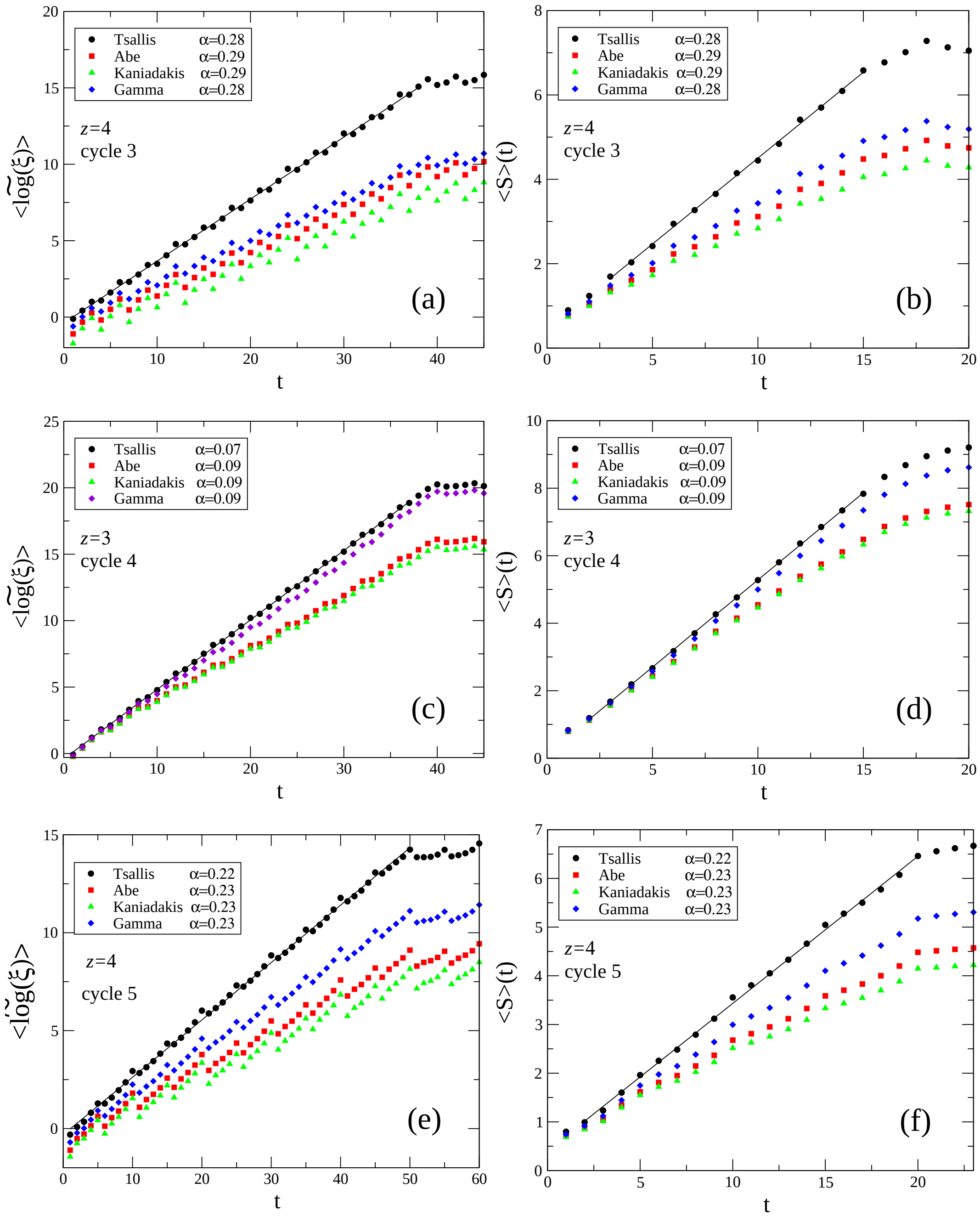


Fig. 1. Time evolution of the sensitivity function averaged over $4 \times 10^7$ uniformly distributed initial conditions for (a) cycle 3 and $z = 4$, (c) cycle 4 and $z = 4$, (e) cycle 5 and $z = 4$. Entropy increase rates with $W = 10^5$ and $N = 10W$ are given in (b),(d) and (f) for the same cycle and $z$ values chosen for the sensitivity function. In each case an average over $W/2$ boxes are performed.

of the sensitivity function become to be more stronger. This is even very much stronger in Kaniadakis case than other three cases. Due to these fluctuations, we did not give any value for $z = 3, 4$ cases of cycle 2 in Table 2. A visualization of these fluctuations is given in Fig. 2. The reason for this is the following : If a given initial condition has weak insensitivity then $\xi$ value is smaller than 1,

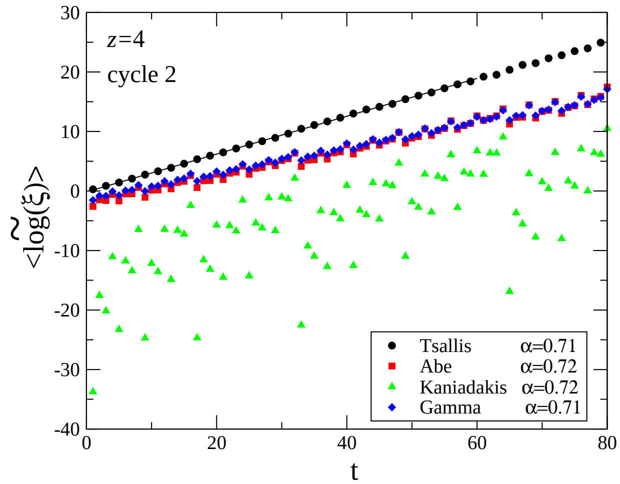


Fig. 2. (a) Time evolution of the sensitivity function averaged over $4 \times 10^7$ uniformly distributed initial conditions for cycle 2 and $z = 4$.

which makes the value of each generalized logarithm negative. But, whenever the $\xi$ value is closer to 0, then logarithms start to go very large negative values and the most rapidly diverging case is the Kaniadakis one. As a result, these very large negative values dominate the sum for the average which yield large fluctuations.

Summing up, in this work, we have numerically analysed the sensitivity to initial conditions and entropy increase rates of $z$-logistic map family at the edge of chaos for various cycles. We have discussed the same generalized logarithms and their related entropies studied in [13] for the standard logistic map and shown that the framework is similar to that of the standard logistic map; namely, (i) the proper $\alpha$ value, which gives a linear time dependence of the sensitivity function and a linear entropy production, is the same for all four entropic forms for a given $z$ value of a given cycle, (ii) the results corroborate a generalized Pesin-like identity $K = \lambda$ for each entropic form with a different numerical value.

## 4 Acknowledgements

One of us (UT) would really like to express his acknowledgements to A. Robledo for his continual constructive comments and encouragement on the subject throughout many years. This work is supported by TUBITAK (Turkish agency) under the research project 104T148.

Table 2
For the generalized logarithms given in the text, three important quantities are listed for cycles 2, 3, 4 and 5 : (i) the appropriate $\alpha$ value which gives a linear time dependence of $\left\langle \widetilde{\log}(\xi) \right\rangle(t)$, (ii) the value of the generalized Lyapunov exponent $\lambda$ which is the slope of the above-mentioned linear time dependence and (iii) the value of entropy increase rates $K$ which is calculated from the slope of the linearly increased entropies coming from each of the generalized logarithms. For many cases, the error bars are $\pm 0.01$ and in a few cases, they are found as $\pm 0.02$.

| cycle | $z$ | Tsallis | | | Abe | | | Kaniadakis | | | $\gamma$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $\alpha$ | $K$ | $\lambda$ | $\alpha$ | $K$ | $\lambda$ | $\alpha$ | $K$ | $\lambda$ | $\alpha$ | $K$ | $\lambda$ |
| 2 | 2 | 0.64 | 0.26 | 0.27 | 0.65 | 0.17 | 0.18 | 0.65 | 0.14 | 0.15 | 0.65 | 0.19 | 0.19 |
| | 2.5 | 0.66 | 0.28 | 0.28 | 0.67 | 0.19 | 0.19 | 0.69 | 0.15 | 0.18 | 0.67 | 0.20 | 0.20 |
| | 3 | 0.67 | 0.28 | 0.28 | 0.67 | 0.18 | 0.18 | −− | −− | −− | 0.67 | 0.19 | 0.19 |
| | 4 | 0.71 | 0.31 | 0.31 | 0.72 | 0.21 | 0.21 | −− | −− | −− | 0.71 | 0.21 | 0.21 |
| 3 | 1.75 | 0.08 | 0.47 | 0.48 | 0.10 | 0.37 | 0.37 | 0.10 | 0.36 | 0.35 | 0.09 | 0.42 | 0.41 |
| | 2 | 0.12 | 0.48 | 0.49 | 0.14 | 0.35 | 0.36 | 0.14 | 0.33 | 0.34 | 0.13 | 0.40 | 0.40 |
| | 2.5 | 0.18 | 0.47 | 0.48 | 0.19 | 0.30 | 0.31 | 0.19 | 0.28 | 0.29 | 0.19 | 0.37 | 0.38 |
| | 3 | 0.22 | 0.44 | 0.45 | 0.23 | 0.29 | 0.29 | 0.23 | 0.26 | 0.26 | 0.23 | 0.35 | 0.35 |
| | 4 | 0.28 | 0.41 | 0.41 | 0.29 | 0.26 | 0.26 | 0.29 | 0.23 | 0.23 | 0.28 | 0.28 | 0.28 |
| 4 | 1.75 | 0.01 | 0.58 | 0.59 | 0.03 | 0.56 | 0.57 | 0.03 | 0.56 | 0.56 | 0.02 | 0.58 | 0.60 |
| | 2 | 0.02 | 0.57 | 0.57 | 0.05 | 0.53 | 0.55 | 0.05 | 0.53 | 0.54 | 0.03 | 0.55 | 0.56 |
| | 2.5 | 0.05 | 0.56 | 0.58 | 0.07 | 0.47 | 0.47 | 0.07 | 0.46 | 0.45 | 0.06 | 0.51 | 0.52 |
| | 3 | 0.07 | 0.52 | 0.53 | 0.09 | 0.41 | 0.41 | 0.09 | 0.40 | 0.39 | 0.09 | 0.48 | 0.50 |
| | 4 | 0.11 | 0.46 | 0.46 | 0.13 | 0.34 | 0.34 | 0.13 | 0.32 | 0.32 | 0.13 | 0.40 | 0.42 |
| 5 | 1.75 | 0.04 | 0.40 | 0.41 | 0.06 | 0.35 | 0.35 | 0.06 | 0.34 | 0.34 | 0.05 | 0.37 | 0.38 |
| | 2 | 0.07 | 0.39 | 0.42 | 0.09 | 0.32 | 0.32 | 0.09 | 0.31 | 0.31 | 0.08 | 0.35 | 0.36 |
| | 2.5 | 0.12 | 0.37 | 0.38 | 0.14 | 0.28 | 0.28 | 0.14 | 0.26 | 0.26 | 0.13 | 0.31 | 0.31 |
| | 3 | 0.16 | 0.35 | 0.35 | 0.18 | 0.25 | 0.25 | 0.18 | 0.23 | 0.24 | 0.17 | 0.28 | 0.28 |
| | 4 | 0.22 | 0.30 | 0.30 | 0.23 | 0.19 | 0.19 | 0.23 | 0.18 | 0.18 | 0.23 | 0.23 | 0.23 |

## References


[1] R.C. Hilborn, *Chaos and Nonlinear Dynamics*, (Oxford University Press, New York , 1994).

[2] V. Latora and M. Baranger, Phys. Rev. Lett. **82** (1999) 520.

[3] C. Tsallis, A.R. Plastino, W.-M. Zheng, Chaos, Solitons and Fractals **8** (1997) 885.

[4] V. Latora, M. Baranger, A. Rapisarda and C. Tsallis, Phys. Lett. A **273** (2000) 97.

[5] C. Tsallis, J. Stat. Phys. **52** (1988) 479.

[6] U.M.S. Costa, M.L. Lyra, A.R. Plastino and C. Tsallis, Phys. Rev. E **56** (1997) 245.

[7] U. Tirnakli, G.F.J. Ananos and C. Tsallis, Phys. Lett. A **289** (2001) 51.

[8] U. Tirnakli, Phys. Rev. E **66** (2001) 066212.

[9] U. Tirnakli, C. Tsallis and M.L. Lyra, Phys. Rev. E **65** (2002) 036207.

[10] F. Baldovin, A. Robledo, Phys. Rev. E **66** (2002) 045104; F. Baldovin, A. Robledo, Europhys. Lett. **60** (2002) 066212; Phys. Rev. E **69** (2004) 045202(R).

[11] G.F.J. Ananos and C. Tsallis, Phys. Rev. Lett. **93** (2004) 020601.

[12] U. Tirnakli and C. Tsallis, Phys. Rev. E **73** (2006), in press.

[13] R. Tonelli et al., Prog. Theor. Phys. **115** (2006) 23.

[14] G. Kaniadakis, M. Lissia and A.M. Scarfone, Phys. Rev. E **71** (2005) 046128 and references therein.

[15] S. Abe, Phys. Lett. A **224** (1997) 326.

[16] G. Kaniadakis, Physica A **316** (2001) 405.